# Basic Projective Geometry

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## 1 Introduction

During this semester in the Directed Reading Program, my mentor, Max Riestenberg helped me explore groups, group actions, projective geometry and hyperbolic space. Some of the readings that helped in the creation of this paper are C. Stanley Ogilvy's book, Excursions in Geometry [1] and Marc Troyanov's paper entitled On the origin of Hilbert Geometry [2].

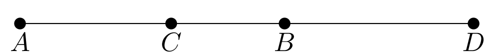
## 2 Projective Geometry

Imagine a Cartesian plane with all the lines that go through the origin. The lines that pass through (0,1) and (0,-1) are considered as having slope infinity. This is considered our line at infinity. Each line that extends from the origin is parallel to one another. Another point on the line could be chosen, so lines that go through this point are parallel to each other.

**Definition 1.** $\mathbb{R}^2$ together with its line at infinity is the projective plane. If two coplanar lines are parallel in the Euclidean sense, their point of intersection lies on the line at infinity.

## 3 The Cross Ratio

**Definition 2.** For a line




the cross ratio is defined

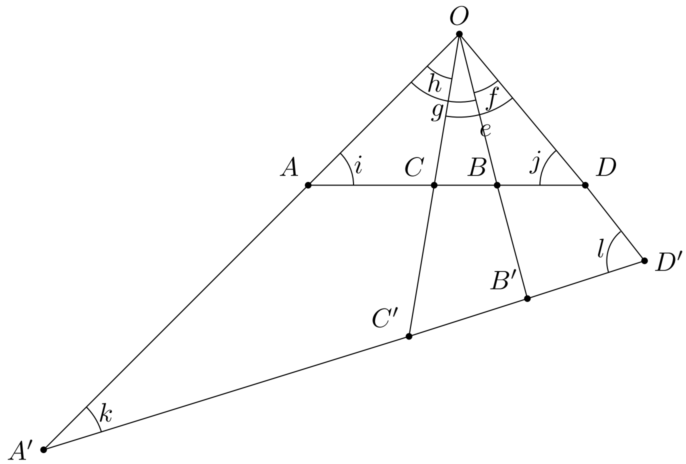
$$[A, D, B, C] = \frac{\overline{DC} \cdot \overline{AB}}{\overline{DB} \cdot \overline{AC}}$$

**Definition 3.** A projection from a point O in ordinary 3-space ($\mathbb{R}^3$) maps the points of a given plane into points of another given plane, as long as O does not lie in either plane. Points and lines are invariant, but angles are not.

**Theorem 1.** The cross ratio is invariant under projection.

*Proof.* Recall the Law of Sines which states that in any triangle the sines of the angles are proportional to the opposite sides:

$$\frac{sin(A)}{a} = \frac{sin(B)}{b} = \frac{sin(C)}{c}$$




We can use the law of sines to find the cross-ratio for $\overline{ACBD}$

$$\frac{sin(e)}{\overline{DC}} = \frac{sin(j)}{\overline{OC}} \implies \frac{\overline{DC}}{sin(e)} = \frac{\overline{OC}}{sin(j)}$$

$$\implies \overline{DC} = \frac{\overline{OC}sin(e)}{sin(j)}$$

$$\frac{\overline{AB}}{sin(g)} = \frac{\overline{OB}}{sin(i)} \implies \overline{AB} = \frac{\overline{OB}sin(g)}{sin(i)}$$

$$\frac{\overline{DB}}{sin(f)} = \frac{\overline{OB}}{sin(j)} \implies \overline{DB} = \frac{\overline{OB}sin(f)}{sin(j)}$$

$$\frac{\overline{AC}}{sin(h)} = \frac{\overline{OC}}{sin(i)} \implies \overline{AC} = \frac{\overline{OC}sin(h)}{sin(i)}$$

We can plug this into the cross ratio

$$[A, D, B, C] = \frac{\overline{DC} \cdot \overline{AB}}{\overline{DB} \cdot \overline{AC}}$$

$$= \frac{\frac{\overline{OC}sin(e)}{sin(j)} \cdot \frac{\overline{OB}sin(g)}{sin(i)}}{\frac{\overline{OB}sin(f)}{sin(j)} \cdot \frac{\overline{OC}sin(h)}{sin(i)}} = \frac{\overline{OC}sin(e) \cdot \overline{OB}sin(g)}{\overline{OB}sin(f) \cdot \overline{OC}sin(h)}$$

$$= \frac{sin(e) \cdot sin(g)}{sin(f) \cdot sin(h)}$$

We can also use the law of sines to find the cross-ratio for $\overline{A'C'B'D'}$

$$\frac{sin(e)}{\overline{D'C'}} = \frac{sin(l)}{\overline{OC'}} \implies \frac{\overline{D'C'}}{sin(e)} = \frac{\overline{OC'}}{sin(l)}$$

$$\implies \overline{D'C'} = \frac{\overline{OC'} sin(e)}{sin(l)}$$

$$\frac{\overline{A'B'}}{sin(g)} = \frac{\overline{OB'}}{sin(k)} \implies \overline{A'B'} = \frac{\overline{OB'} sin(g)}{sin(k)}$$

$$\frac{\overline{D'B'}}{sin(f)} = \frac{\overline{OB'}}{sin(l)} \implies \overline{D'B'} = \frac{\overline{OB'} sin(f)}{sin(l)}$$

$$\frac{\overline{A'C'}}{sin(h)} = \frac{\overline{OC'}}{sin(k)} \implies \overline{A'C'} = \frac{\overline{OC'} sin(h)}{sin(k)}$$

We can plug this into the cross ratio

$$[A', D', B', C'] = \frac{\overline{D'C'} \cdot \overline{A'B'}}{\overline{D'B'} \cdot \overline{A'C'}}$$

$$= \frac{\frac{\overline{OC'} sin(e)}{sin(l)} \cdot \frac{\overline{OB'} sin(g)}{sin(k)}}{\frac{\overline{OB'} sin(f)}{sin(l)} \cdot \frac{\overline{OC'} sin(h)}{sin(k)}} = \frac{\overline{OC'} sin(e) \cdot \overline{OB'} sin(g)}{\overline{OB'} sin(f) \cdot \overline{OC'} sin(h)}$$

$$= \frac{sin(e) \cdot sin(g)}{sin(f) \cdot sin(h)}$$

$$[A', D', B', C'] = [A, D, B, C] = \frac{sin(e) \cdot sin(g)}{sin(f) \cdot sin(h)}$$

Therefore the cross ratio is preserved under projection.

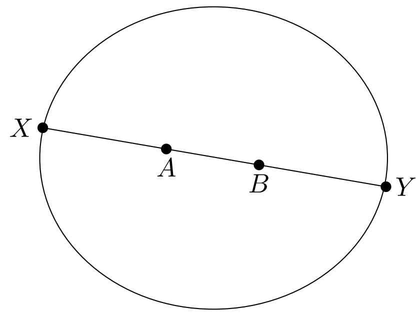
□

# 4 The Hilbert Metric

**Definition 4.** A convex projective domain $\Omega$ is an open, nonempty, convex subset of $\mathbb{RP}^2$ whose closure lies in an affine chart.

The Hilbert metric can be used to define distance on a convex projective domain.

**Definition 5.** (Hilbert metric) For a bounded convex region $\Omega$ in $\mathbb{R}^2$. $A$ and $B$ exist in the region, creating a line where $X$ and $Y$ are on the boundary $\partial\Omega$.




$d(A,B) = \frac{1}{2}\log[X,Y,B,A]$ where $[X,Y,B,A] = \frac{\overline{YA}\cdot\overline{XB}}{\overline{YB}\cdot\overline{XA}}$

**Definition 6.** A metric space is a set $X$ together with a function $d$ which assigns a real number $d(x,y)$ to every pair $x,y \in X$ satisfying

1. $d(x,y) > 0$, and $d(x,y) = 0 \iff x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,y) + d(y,z) \leq d(x,z)$

**Theorem 2.** The Hilbert Metric is a metric space.

*Proof.* 1. Distance is defined by $d(A,B) = \frac{1}{2}\log[X,Y,B,A] = \frac{1}{2}\log\left(\frac{\overline{YA}\cdot\overline{XB}}{\overline{YB}\cdot\overline{XA}}\right)$. Because $\overline{YB} < \overline{YA}$ and $\overline{XA} < \overline{XB}$, it is clear that $\frac{\overline{YA}}{\overline{YB}}, \frac{\overline{XB}}{\overline{XA}} > 1$.

$$\implies d(A,B) = \frac{1}{2}\log\left(\frac{\overline{YA}\cdot\overline{XB}}{\overline{YB}\cdot\overline{XA}}\right) > 0$$

If A=B, then $\overline{XA} = \overline{XB}$ and $\overline{YA} = \overline{YB}$. And therefore $d(A,B) = \frac{1}{2}\log[X,Y,B,A] = \frac{1}{2}\log(\frac{\overline{YA}\cdot\overline{XB}}{\overline{YB}\cdot\overline{XA}}) = \frac{1}{2}\log(1\cdot 1) = 0$

Now assume $d(A,B) = 0$

$$\implies \frac{1}{2}\log[X,Y,B,A]$$

$$\implies \log[X,Y,B,A] = 0$$

$$\implies \frac{\overline{YA}\cdot\overline{XB}}{\overline{YB}\cdot\overline{XA}} = 1$$

$$\implies \overline{XA} = \overline{XB} \wedge \overline{XA} = \overline{XB}$$

or

$$\overline{YB} = \overline{XB} \wedge \overline{YA} = \overline{XB}$$

$$\implies A = B \vee Y = X$$

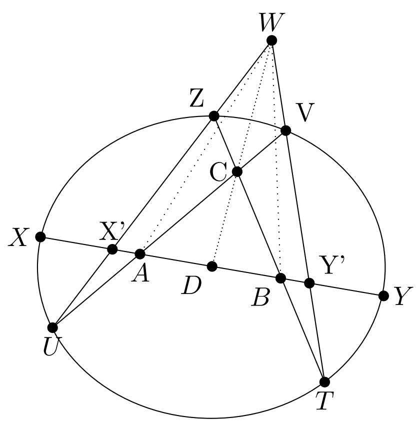
But $A$ and $B$ are assumed to lie between $X$ and $Y$, so if $X$ and $Y$ are equal, $X = Y = A = B$. Therefore $A = B$.

2.

$$d(B, A) = \frac{1}{2}\log[Y, X, A, B] = \frac{1}{2}\log\left(\frac{\overline{XB}\cdot\overline{YA}}{\overline{XA}\cdot\overline{YB}}\right)$$

$$= \frac{1}{2}\log\left(\frac{\overline{YA}\cdot\overline{XB}}{\overline{YB}\cdot\overline{XA}}\right) = \frac{1}{2}\log[X, Y, B, A] = d(A, B)$$

3.




Note: $[U, V, C, A] = [X', Y', D, A]$ and $[Z, T, B, C] = [X', Y', B, D]$

$$d(C, A) + d(C, B) = \frac{1}{2}\log[U, V, C, A] + \frac{1}{2}\log[Z, T, B, C]$$

$$= \frac{1}{2}\log[X', Y', D, A] + \frac{1}{2}\log[X', Y', B, D]$$

$$= \frac{1}{2}\log([X', Y', D, A]\cdot[X', Y', B, D])$$

$$= \frac{1}{2}\log\left(\frac{\overline{Y'A}\cdot\overline{X'D}}{\overline{Y'D}\cdot\overline{X'A}}\right)\cdot\left(\frac{\overline{Y'D}\cdot\overline{X'B}}{\overline{Y'B}\cdot\overline{X'D}}\right)$$

$$= \frac{1}{2}\log\left(\frac{\overline{Y'A}\cdot\overline{X'D}\cdot\overline{Y'D}\cdot\overline{X'B}}{\overline{Y'D}\cdot\overline{X'A}\cdot\overline{Y'B}\cdot\overline{X'D}}\right)$$

$$= \frac{1}{2}\log\left(\frac{\overline{Y'A}\cdot\overline{X'B}}{\overline{X'A}\cdot\overline{Y'B}}\right) = \frac{1}{2}\log[X', Y', B, A]$$

So,

$$d(C, A) + d(C, B) = \frac{1}{2} \log[X', Y', B, A]$$

The distance $d(A, B)$ depends on the convex domain, so $d(A, B)$ becomes greater as $X$ approaches $A$ and $Y$ approaches $B$. This means the following equality is true.

$$\frac{1}{2} \log[X', Y', B, A] \geq \frac{1}{2} \log[X, Y, B, A]$$

Therefore $d(C, A) + d(C, B) \geq d(A, B)$. And the space is a metric space. $\square$

# References


[1] Charles Stanley Ogilvy. *Excursions in geometry*. Courier Corporation, 1990.

[2] Marc Troyanov-EPFL. On the origin of hilbert geometry. *arXiv preprint arXiv:1407.3777*, 2014.